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Structures in BC_N Ruijsenaars-Schneider models

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ABSTRACT

We construct the classical r -matrix structure for the Lax formulation of BC_N Ruijsenaars-Schneider systems proposed in [18]. The r -matrix structure takes a quadratic form similar to the A_N Ruijsenaars-Schneider Poisson bracket behavior, although the dynamical dependence is more complicated. Commuting Hamiltonians stemming from the BC_N Ruijsenaars-Schneider Lax matrix are shown to be linear combinations of particular Koornwinder-van Diejen “external fields” Ruijsenaars-Schneider models, for specific values of the exponential one-body couplings. Uniqueness of such commuting Hamiltonians is established once the first of them and the general analytic structure are given.

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1 Introduction

The relativistic extensions of Calogero-Moser N -body integrable systems, originally introduced by Ruijsenaars and Schneider [1], have been the subject of numerous investigations in these last years. Their exact connection to field-theoretical integrable systems, initially described in [1], was clarified in [2]; their dynamical classical r -matrix structure, first tackled in [3, 4] was finally established in [5] and characterized as a quadratic structure, à la Sklyanin [6, 7], stemming from the dynamical linear r -matrix structure of Calogero-Moser systems [8]. This in turn is connected to the realization of such systems [9] as “Hamiltonian reduction”, in a more general sense see [10], of dynamical systems living on a Heisenberg double [10, 11], where the quadratic r -matrix structure is natural [10]. More complete descriptions may be found in [13].

More recently several extensions of the Ruijsenaars-Schneider systems were considered. First of all, one-body-potential (or “external field extensions”) were constructed in [14]. The integrability proof, and construction of the quantum Hamiltonians themselves, rested upon polynomial-algebraic arguments [15] pioneered by Koornwinder [16] and indicating a connection to BC_N -type algebras. Both Lax formulation and classical r -matrix construction were lacking. Quantum elliptic van Diejen-type Hamiltonians were then constructed in [17] using corner-transfer-matrix methods, and incidentally pointing again to a BC_N structure underlying at least some particular Koornwinder-van Diejen potentials. The corner-transfer-matrix method used in [17] however does not exhibit a clear-cut way of defining a classical limit with one single Lax matrix, using as it does two types of Lax operators. Finally an explicit construction of classical BC_N and C_N Ruijsenaars-Schneider models was presented in [18], using a \mathbb{Z}_2 -folding of the original A_{2N} or A_{2N+1} algebra intrinsically present in original Ruijsenaars-Schneider models. The Lax matrix and classical commuting Hamiltonians were then constructed, again as \mathbb{Z}_2 -foldings of the A_{2N} (A_{2N+1}) Lax matrix and commuting Hamiltonians. The elliptic BC_N and C_N systems were then defined in [19], and the D_N case was considered in [20].

This series of results still begs several questions and we wish to address here two points which remained unclarified.

First of all we construct the classical r -matrix for hyperbolic BC_N models (we expect that the C_N and D_N case, and the elliptic potentials, may be treated by similar techniques although the elliptic case may endow more complicated algebraic manipulations). This problem may seem academic, since in any case commutation of the Hamiltonians follows from the construction itself [18]. However it actually sheds light on the delicate question of interplay between the folding procedure $A_{2N} \rightarrow BC_N$ and the initial “quadratization” of Poisson structure entailed by the change of base symplectic manifold for the “Hamiltonian” reduction from T^*g (cotangent bundle Lie group) to Dg_H (Heisenberg double of Lie group)¹. Indeed it will eventually turn out

¹Similar difficulties occur when considering the quantum deformation of BC - or D -type algebras, compared with deformation of A_N algebras. This fact was pointed out to us by D. Arnaudon.

that the classical r -matrix for hyperbolic Ruijsenaars-Schneider BC_N models may be recast under a quadratic form, similar to the A_N case, but contrary to what occurred in the Calogero-Moser models [21] the structure (after folding) exhibits now a dependance in both sets of conjugated dynamical variables, and the quadratic r -matrix is thus not directly related to the linear dynamical r -matrix structure for BC_N Calogero-Moser models.

The second problem which we consider here deals with the connection between the initial Koornwinder-van Diejen Hamiltonians and the “canonical” Poisson commuting Hamiltonians generated by the traces of powers of the Lax matrix for BC_N systems. It will be shown that the Koornwinder-van Diejen Hamiltonians are in fact combinations of the “canonical ones”, and this property is actually true for any set of Poisson-commuting Hamiltonians with the same functional structure (to be explicated hereafter).

2 The classical r -matrix structure

2.1 The BC_N Ruijsenaars-Schneider models and notations

The canonical variables are a set of rapidities $\{\theta_i, i = 1 \cdots N\}$ and conjugate positions q_i such that $\{\theta_i, q_j\} = \delta_{ij}$. The Hamiltonian reads:

$$H = \sum_{\substack{j=1 \\ \varepsilon = \pm 1}}^N e^{-\varepsilon \beta \theta_j} f_j + \mathcal{U} \quad (1)$$

$$\text{where } f_j = \left[f(q_j) f(2q_j) \prod_{\substack{k=1, \\ k \neq j}}^N f(q_j - q_k) f(q_j + q_k) \right]^{1/2} \quad \text{and} \quad \mathcal{U} = \prod_{k=1}^N f(q_k).$$

Function f may take different forms, namely:

$$\begin{aligned} f(q) &= 1 - \frac{g^2}{q^2} \text{ (rational)} \\ f(q) &= 1 - \frac{\sinh^2 \gamma}{\sinh^2 \frac{\nu q}{2}} \text{ (hyperbolic)} \\ f(q) &= 1 - \frac{\sin^2 \gamma}{\sin^2 \frac{\nu q}{2}} \text{ (trigonometric).} \end{aligned}$$

The most general elliptic case where:

$$f(q) = (\lambda + \nu \mathcal{P}(q)) \text{ , } \mathcal{P} = \text{Weierstrass function}$$

will not yet be considered here.

Since the trigonometric and hyperbolic cases define the same model at least locally up to a redefinition of the parameters ², and the rational case is obtained by an easy limit procedure from one of these models, we shall consider in the following only the hyperbolic version.

Let us note that one can also write $f(q) = v(q) v(-q)$, with :

$$v(q) = \frac{\sinh(\frac{\nu q}{2} + \gamma)}{\sinh \frac{\nu q}{2}}$$

or even as a rational fonction of an exponential variable:

$$v(q) = \lambda^{-1/2} \frac{z - \lambda}{z - 1} \quad \text{with} \quad z = e^{\nu q} \quad \text{and} \quad \lambda = e^{-2\gamma}.$$

This rational formulation will be useful to establish some functional identities à la Liouville [9].

2.2 The BC_N Lax operator

As shown in [18], the Lax formulation of BC_N Ruijsenaars-Schneider system may be obtained as a folding of the A_{2n} case. The reduction works as follow: labelling the $2N + 1$ rapidities $\{\theta_i, i = -N \cdots N\}$ and conjugate positions $\{q_i, i = -N \cdots N\}$, one sets $\theta_i = \varepsilon_i \theta_{|i|}$ and $q_i = \varepsilon_i q_{|i|}$ with:

$$\varepsilon_i = \begin{cases} +1 & \text{for } 1 \leq i \leq N \\ 0 & \text{for } i = 0 \\ -1 & \text{for } -1 \geq i \geq -N \end{cases}.$$

The Lax matrix for the A_{2n} cases is:

$$\begin{aligned} \mathcal{L} &= \sum_{i,j=-N}^N \mathcal{L}_{ij} e_{ij} \\ \mathcal{L}_{ij}(q_1, \dots, q_N, q_0, q_{-1}, \dots, q_{-N}, \theta_j) &= c(q_i - q_j) e^{-\beta \theta_j} \prod_{\substack{k=-N, \\ k \neq j}}^N f^{1/2}(q_j - q_k) \end{aligned} \quad (2)$$

where $\{e_{ij}\}$ is the standard basis for $(2N + 1) \times (2N + 1)$ matrices; f was given in the previous subsection and

$$c(q) = \frac{\sinh \gamma}{\sinh(\frac{\nu q}{2} + \gamma)} = (1 - \lambda) \frac{z^{1/2}}{z - \lambda}.$$

² The global structure of trigonometric vs hyperbolic models is however quite different, owing to topological properties, as can be seen for instance in [23]

The Lax matrix for the BC_N Ruijsenaars-Schneider systems then reads:

$$L = \sum_{i,j=-N}^N L_{ij} e_{ij} \quad \text{with} \quad L_{ij} = \mathcal{L}_{ij}(q_1, \dots, q_N, 0, -q_1, \dots, -q_N, \varepsilon_j \theta_{|j|}) \quad (3)$$

It can be rewritten: $L_{ij} = c(q_i - q_j) e^{-\beta \varepsilon_j \theta_{|j|}} f_j$, extending the definition of f_j given in (1) to $j \in \{-N \dots N\}$.

Note that with this extension of f_j one has $f_0 = \mathcal{U}$ and $f_j = f_{-j}$.

It has been shown that the Lax operator (2) satisfies the quadratic fundamental Poisson bracket [5]:

$$\{\mathcal{L} \otimes \mathcal{L}\} = \mathcal{L} \otimes \mathcal{L} a_1 - a_2 \mathcal{L} \otimes \mathcal{L} + \mathcal{L}_2 s_1 \mathcal{L}_1 - \mathcal{L}_1 s_2 \mathcal{L}_2, \quad (4)$$

where $\mathcal{L}_1 = \mathcal{L} \otimes 1$, $\mathcal{L}_2 = 1 \otimes \mathcal{L}$ and the quadratic structure coefficients read:

$$\begin{aligned} a_1 &= a + w, & s_1 &= s - w, \\ a_2 &= a + s - s^\pi - w, & s_2 &= s^\pi + w. \end{aligned}$$

For any matrix M , the matrix M^π is defined by:

$$\text{if } M \equiv \sum_{ijkl=-N}^N M_{ijkl} e_{ij} \otimes e_{kl} \quad \text{then } M^\pi = \sum_{ijkl=-N}^N M_{ijkl} e_{kl} \otimes e_{ij}.$$

Matrices a, s, w take the form:

$$\begin{aligned} a &= \alpha \sum_{\substack{j, k = -N \\ k \neq j}}^N \coth \frac{\nu}{2} (q_k - q_j) e_{jk} \otimes e_{kj}, \\ s &= -\alpha \sum_{\substack{j, k = -N \\ k \neq j}}^N \frac{1}{\sinh \frac{\nu}{2} (q_k - q_j)} e_{jk} \otimes e_{kk}, \\ w &= \alpha \sum_{\substack{j, k = -N \\ k \neq j}}^N \coth \frac{\nu}{2} (q_k - q_j) e_{kk} \otimes e_{jj} \end{aligned} \quad (5)$$

where $\alpha \equiv \beta \frac{\nu}{2}$.

It must be recalled here that the most general structure of Poisson bracket for a Lax operator of a Liouville-integrable system is a linear one [22]:

$$\{L \otimes L\} = [r, L_1] - [r^\pi, L_2]. \quad (6)$$

The quadratic form (4) corresponds to the case where the r -matrix itself assumes a linear dependency in L of type:

$$r = b L_2 + L_2 c, \quad (7)$$

with b and c arbitrary matrices determining the quadratic coefficients a_1, a_2, s_1, s_2 :

$$a_1 = c^\pi - c, \quad a_2 = b^\pi - b, \quad s_1 = c + b^\pi \quad \text{and} \quad s_2 = s_1^\pi.$$

In the next subsection, we will show that the BC_N Ruijsenaars-Schneider Lax operator (3) also satisfies a quadratic fundamental Poisson bracket (4) albeit with a fundamental difference with respect to (5) regarding the dependence on the dynamical variables. We will give explicitly the generalizations of the matrices a_1, a_2, s_1 and s_2 , hereafter denoted “quadratic r -matrices” for obvious semantic reasons.

2.3 Computation of the classical r -matrix

Let us calculate the Poisson brackets of the Lax matrix (3):

$$\{L_{ij}, L_{kl}\} = \beta L_{ij} L_{kl} \left(\varepsilon_l \frac{\partial \ln L_{ij}}{\partial q_{|l|}} - \varepsilon_j \frac{\partial \ln L_{kl}}{\partial q_{|j|}} \right)$$

and express it in terms of the Lax matrix (2),

$$\varepsilon_l \frac{\partial \ln L_{ij}}{\partial q_{|l|}} = \left(\frac{\partial \ln \mathcal{L}_{ij}}{\partial q_l} - \frac{\partial \ln \mathcal{L}_{ij}}{\partial q_{-l}} \right).$$

We thus get:

$$\{L_{ij}, L_{kl}\} = \{\mathcal{L}_{ij}, \mathcal{L}_{kl}\} + \beta L_{ij} L_{kl} \left(\frac{\partial \ln \mathcal{L}_{kl}}{\partial q_{-j}} - \frac{\partial \ln \mathcal{L}_{ij}}{\partial q_{-l}} \right).$$

The Poisson bracket of the first term on r.h.s. keeps the same form (4) where one should fold the dynamical variables ($\theta_i = \varepsilon_i \theta_{|i|}$ and $q_i = \varepsilon_i q_{|i|}$). We thus only need to concentrate on the remaining term, introducing the four-index object:

$$U_{ijkl} \equiv \frac{2}{\nu} \left(\frac{\partial \ln \mathcal{L}_{kl}}{\partial q_{-j}} - \frac{\partial \ln \mathcal{L}_{ij}}{\partial q_{-l}} \right).$$

Straightforward calculations yield:

$$U_{ijkl} = \delta_{j,-l} u_j + (\delta_{i,-l} - \delta_{j,-l}) t_{ij} - (\delta_{j,-k} - \delta_{j,-l}) t_{kl}$$

where

$$\begin{aligned} t_{ij} &= -\frac{2}{\nu} (\ln c)'(q_i - q_j) = \coth\left(\frac{\nu}{2}(q_i - q_j) + \gamma\right) = \frac{z_i + \lambda z_j}{z_i - \lambda z_j} \\ u_j &= \frac{2}{\nu} \sum_{\substack{k=-N, \\ k \neq j}}^N (\ln f)'(q_k - q_j) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{k=-N, \\ k \neq j}}^N 2 \coth \frac{\nu}{2}(q_j - q_k) + \coth \left(\frac{\nu}{2}(q_k - q_j) + \gamma \right) - \coth \left(\frac{\nu}{2}(q_j - q_k) + \gamma \right) \\
&= \sum_{\substack{k=-N, \\ k \neq j}}^N 2 \frac{z_j + z_k}{z_j - z_k} + \frac{z_k + \lambda z_j}{z_k - \lambda z_j} - \frac{z_j + \lambda z_k}{z_j - \lambda z_k} = \sum_{\substack{k=-N, \\ k \neq j}}^N 2 a_{jk} + t_{kj} - t_{jk},
\end{aligned}$$

with $a_{jk} = \coth \frac{\nu}{2}(q_j - q_k) = \frac{z_j + z_k}{z_j - z_k}$ for $j \neq k$.

Note the following properties of these objects on the folded space:

$$t_{-j-i} = t_{ij}, \quad a_{-j-i} = a_{ij}, \quad \text{and} \quad u_{-i} = -u_i.$$

In the expression of U_{ijkl} , the terms in $\delta_{j,-l}$ are not on the same footing as the others since they are separately antisymmetric under the exchange of the two spaces (operation π) whereas the remaining terms verify this property only altogether.

We thus first take care of the $\delta_{j,-l}$ terms, introducing the matrix ρ :

$$\rho = \alpha \sum_{k,l=-N}^N L_{kl} \left(\frac{1}{2} u_l - t_{kl} \right) e_{-l-l} \otimes e_{kl},$$

realizing them as a linear r -matrix form (6):

$$\begin{aligned}
([\rho, L_1] - [\rho^\pi, L_2])_{ijkl} &= \alpha L_{ij} L_{kl} \left(\delta_{j,-l} (u_j - t_{ij} + t_{kl}) + \delta_{i,-l} \left(\frac{1}{2} u_l - t_{kl} \right) - \delta_{j,-k} \left(\frac{1}{2} u_j - t_{ij} \right) \right) \\
&= \alpha L_{ij} L_{kl} (U_{ijkl} - \tilde{U}_{ijkl}), \quad \text{with} \quad \tilde{U}_{ijkl} = \delta_{i,-l} (t_{ij} + t_{kl} - \frac{1}{2} u_l) - \delta_{j,-k} (t_{ij} + t_{kl} - \frac{1}{2} u_j).
\end{aligned}$$

We may furthermore bring it back to our seeked-for general quadratic form by setting $\rho = \tau L_2$ (i.e. taking $b = \tau$ and $c = 0$ in (7)), since the matrix L is invertible:

$$\tau = \rho L_2^{-1} = \alpha \sum_{i,k,l=-N}^N L_{k-i} L_{-il}^{-1} \left(\frac{1}{2} u_{-i} - t_{k-i} \right) e_{ii} \otimes e_{kl}. \quad (8)$$

One should immediately note, from the explicit form of L (3), that this matrix τ actually does not depend on the rapidities θ_i 's. We are therefore still in the “canonical” quadratic structure (a la Suris) of type (4) with dynamical quadratic r -matrices depending only on one set of canonical variables (the q_i 's).

We will now show that there exists a matrix σ , such that $\sigma^\pi = \sigma$ and:

$$\alpha L_{ij} L_{kl} \tilde{U}_{ijkl} = [L_2 \sigma L_1 - L_1 \sigma^\pi L_2]_{ijkl}.$$

This corresponds to setting $c = \sigma$ and $b = 0$ in (7) and thus actually formally completes the quadratic r -matrix structure (4).

In order to ensure self-consistency of the dependence in the indices, we assume the following tensorial structure:

$$\sigma = \sum_{m,n=-N}^N \sigma_{mn} e_{mn} \otimes e_{-n-m}, \quad \text{satisfying} \quad \sigma^\pi = \sigma \quad \text{ie:} \quad \sigma_{-n-m} = \sigma_{mn},$$

yielding:

$$[L_2 \sigma L_1 - L_1 \sigma^\pi L_2]_{ijkl} = \delta_{i,-l} \sum_{n=-N}^N \sigma_{in} L_{k-n} L_{nj} - \delta_{j,-k} \sum_{n=-N}^N \sigma_{kn} L_{i-n} L_{nl}.$$

The set of equations to be solved then reads:

$$\delta_{i,-l} \left(\sum_{n=-N}^N \sigma_{in} \frac{L_{k-n} L_{nj}}{L_{ij} L_{kl}} - \alpha \left(t_{ij} + t_{kl} - \frac{1}{2} u_l \right) \right) = \delta_{j,-k} \left(\sum_{n=-N}^N \sigma_{kn} \frac{L_{i-n} L_{nl}}{L_{ij} L_{kl}} - \alpha \left(t_{ij} + t_{kl} - \frac{1}{2} u_j \right) \right),$$

or, equivalently:

$$\sum_{n=-N}^N \sigma_{in} \frac{L_{k-n} L_{nj}}{L_{ij} L_{k-i}} - \alpha \left(t_{ij} + t_{k-i} + \frac{1}{2} u_i \right) = \delta_{j,-k} s_{ik}, \quad \text{with} \quad s_{ik} = s_{ki}.$$

Direct calculations yield:

$$\frac{L_{k-n} L_{nj}}{L_{ij} L_{k-i}} = \frac{e^{\beta \theta_n} f_n}{e^{\beta \theta_i} f_i} \left((1 - \delta_{j,-k}) \frac{t_{nj} - t_{n-k}}{t_{ij} - t_{i-k}} + \delta_{j,-k} \frac{t_{nj}^2 - 1}{t_{ij}^2 - 1} \right),$$

and equations become:

$$\begin{aligned} \sum_{n=-N}^N \tilde{\sigma}_{in} (t_{nj} - t_{n-k}) &= \alpha (t_{ij}^2 - t_{i-k}^2 + \frac{1}{2} u_i t_{ij} - \frac{1}{2} u_i t_{i-k}) \\ \sum_{n=-N}^N \tilde{\sigma}_{in} \frac{t_{nj}^2 - 1}{t_{ij}^2 - 1} - \alpha (2 t_{ij} + \frac{1}{2} u_i) &= s_{i-j}, \quad \text{with} \quad \tilde{\sigma}_{in} = \frac{e^{\beta \theta_n} f_n}{e^{\beta \theta_i} f_i} \sigma_{in}. \end{aligned}$$

Let us recall here that the only additional constraints on matrices $\tilde{\sigma}$ and s read:

$$\begin{cases} \tilde{\sigma}_{-j-i} = \frac{f_i^2}{f_j^2} \tilde{\sigma}_{ij} \\ s_{ij} = s_{ji} \end{cases} \quad (9)$$

According to the previous equations, the matrix s is determinated given the matrix $\tilde{\sigma}$, and $\tilde{\sigma}$ is obtained, up to a one-dimensional degree of freedom v_i , by:

$$\sum_{n=-N}^N \tilde{\sigma}_{in} t_{nj} = \alpha (t_{ij}^2 + \frac{1}{2} u_i t_{ij} + v_i), \quad (10)$$

since t is invertible.

Remains to verify that one can find v_i 's, such that the compatibility relations (9) be satisfied, namely:

$$\sum_{m=-N}^N t_{im} f_m^2 t_{mj} (t_{mj} - t_{im} + u_m) = \sum_{m=-N}^N f_m^2 (v_{-m} t_{mj} - v_m t_{im}) \quad (11)$$

$$(v_i + 1) \sum_{n,m=-N}^N (t_{-jm}^2 - 1) (t^{-1})_{mn} = (v_{-j} + 1) \sum_{n,m=-N}^N (t_{im}^2 - 1) (t^{-1})_{mn} \quad (12)$$

Equation (12) directly yields $v_i = -1 + \eta \sum_{n,m=-N}^N (t_{im}^2 - 1) (t^{-1})_{mn}$, with η an arbitrary constant.

In order to solve (11) we shall first compute its left hand side.

We do so, by twofold evaluation of the following contour integral in the complex plane around infinity:

$$I_{ij} = \frac{1}{2\pi i} \oint_{C_\infty} \frac{dz}{z} \frac{z_i + \lambda z}{z_i - \lambda z} \frac{z + \lambda z_j}{z - \lambda z_j} \prod_{k=-N}^N \frac{z - \lambda z_k}{z - z_k} \frac{z_k - \lambda z}{\lambda(z_k - z)}.$$

Contour C_∞ is oriented counterclockwise and loops around infinity.

Residue at infinity gives $I_{ij} = -1$.

Whereas I_{ij} also equals the sum of residues at poles of the meromorph integrand in the whole complex plane, that is: a single pole at $z = 0$ with residue -1 and a set of double poles at $z = z_m$. We thus obtain:

$$\sum_{m=-N}^N \left[\frac{z_i + \lambda z}{z_i - \lambda z} \frac{z + \lambda z_j}{z - \lambda z_j} \frac{(z - \lambda z_m)(z_m - \lambda z)}{-\lambda z} \prod_{\substack{k=-N, \\ k \neq m}}^N \frac{z - \lambda z_k}{z - z_k} \frac{z_k - \lambda z}{\lambda(z_k - z)} \right]' (z = z_m) = 0.$$

Noticing that:

$$\begin{aligned} u_m &= \frac{2}{\nu} \sum_{\substack{k=-N, \\ k \neq m}}^N (\ln f)'(q_k - q_m) = -\frac{2}{\nu} \frac{\partial}{\partial q_m} \ln \prod_{\substack{k=-N, \\ k \neq m}}^N f(q_k - q_m) \\ &= -2 z_m \frac{\partial}{\partial z_m} \ln \prod_{\substack{k=-N, \\ k \neq m}}^N \frac{z_m - \lambda z_k}{z_m - z_k} \frac{z_k - \lambda z_m}{\lambda(z_k - z_m)} \\ &= -2 z_m \left[\ln \prod_{\substack{k=-N, \\ k \neq m}}^N \frac{z - \lambda z_k}{z - z_k} \frac{z_k - \lambda z}{\lambda(z_k - z)} \right]' (z = z_m) \end{aligned}$$

and also:

$$z_m \left[\ln \left(\frac{z_i + \lambda z}{z_i - \lambda z} \frac{z + \lambda z_j}{z - \lambda z_j} \right) \right]' (z = z_m) = \frac{1}{2} ((t_{im} - t_{mj}) - (\frac{1}{t_{im}} - \frac{1}{t_{mj}})),$$

we obtain:

$$\sum_{m=-N}^N t_{im} f_m^2 t_{mj} \left[t_{mj} - t_{im} + u_m + (\frac{1}{t_{im}} - \frac{1}{t_{mj}}) \right] = 0,$$

or equivalently:

$$\sum_{m=-N}^N t_{im} f_m^2 t_{mj} (t_{mj} - t_{im} + u_m) = \sum_{m=-N}^N f_m^2 (t_{im} - t_{mj}).$$

This derivation of Liouville-type functional identities stems from similar derivations to be found in [9].

Equation (11) now reads:

$$\sum_{m=-N}^N t_{im} f_m^2 (v_m + 1) = \sum_{m=-N}^N (v_{-m} + 1) f_m^2 t_{mj} = \sum_{m=-N}^N t_{-jm} f_m^2 (v_m + 1),$$

and can be solved straightforwardly: $v_i = -1 + \xi \frac{1}{f_i^2} \sum_{m=-N}^N (t^{-1})_{im}$, with ξ any constant.

We thereby prove the consistency of (9) and (10), since $v_i = -1$ is an obvious solution (with $\eta = \xi = 0$).

In addition, performing calculations of the same type of contour integrals, one gets:

$$\sum_{n,m=-N}^N (t_{im}^2 - 1) (t^{-1})_{mn} = -\frac{4\lambda}{(\lambda-1)^2} \frac{1}{f_i^2} \sum_{m=-N}^N (t^{-1})_{im} \propto \frac{1}{D_i^+},$$

with $D_i^+ = \prod_{\substack{k=-N, \\ k \neq i}}^N \frac{z_i - \lambda z_k}{z_i - z_k}.$

The two forms which the v_i 's should satisfy, are actually identical: $v_i = -1 + \frac{\zeta}{D_i^+}$, ζ being an arbitrary constant.

We fix this remaining gauge, setting $\zeta = \frac{1 - \lambda^{2N+1}}{1 - \lambda}$, in order to obtain the simplest form for $\tilde{\sigma}$ from relation (10):

$$\tilde{\sigma}_{ij} = \alpha \frac{D_j^+}{D_i^+} (\delta_{i,j} s_i - (1 - \delta_{i,j}) a_{ij}), \quad \text{where} \quad s_i = \frac{1 + \lambda}{1 - \lambda} + \sum_{m=-N}^N \frac{1}{2} (t_{mi} + t_{im}).$$

We finally give the expression of the matrix σ :

$$\sigma = \alpha \sum_{i,j=-N}^N \frac{\mathcal{A}_j}{\mathcal{A}_i} (\delta_{i,j} s_i - (1 - \delta_{i,j}) a_{ij}) e_{ij} \otimes e_{-j-i}, \quad \text{with} \quad \mathcal{A}_i = \sqrt{\frac{D_i^+}{D_{-i}^+}} e^{-\beta \theta_i}. \quad (13)$$

The r -matrix structure is now completely defined by a quadratic Poisson bracket of type (4) where the quadratic r -matrices a_1 , a_2 , s_1 and s_2 are changed into:

$$\begin{aligned} a_1 &\rightarrow \tilde{a}_1 = a_1, \\ s_1 &\rightarrow \tilde{s}_1 = s_1 + \sigma + \tau^\pi \\ s_2 &\rightarrow \tilde{s}_2 = \tilde{s}_1^\pi = s_2 + \sigma^\pi + \tau = s_2 + \sigma + \tau \\ a_2 &\rightarrow \tilde{a}_2 = \tilde{a}_1 + \tilde{s}_1 - \tilde{s}_2 = a_2 + \tau^\pi - \tau, \end{aligned} \quad (14)$$

and matrices σ and τ are respectively defined by equations (13) and (8).

2.4 Comments

One should notice that this quadratic r -matrix structure is now fully dynamical, depending both on the positions q_i 's and rapidities θ_i 's. Moreover, its conjugating factor \mathcal{A}_i , which bears this double dependance, is deeply connected to the structure of the matrix L under folding:

$$L_{ij}^{-1} = \frac{\mathcal{A}_j}{\mathcal{A}_i} L_{-i-j}$$

We have here an interesting first example of a “doubly dynamical” r -matrix dependence, stemming from the interplay between the folding procedure leading from A_N to BC_N , and the quadratic structure of Ruijsenaars-Schneider-type models. This seems to open new perspectives, first of all on the classification of such doubly dynamical r -matrices. The only other example known to us at this time is the classical linear r -matrix structure for the Lax formulation of the A_N elliptic Calogero-Moser model in the absence of spectral parameter [24]. Proof of its double dynamical dependence is given in [25] although the explicit form is yet conjectural [26]. Curiously however, it occurs in relation with an A_N model with no folding and may therefore be of a different nature.

Interpretation of doubly dynamical objects in the frame of quantum group theory is yet lacking. “Simply” dynamical r -matrices are known to be connected to the theory of Drinfel'd-twisted quantum groups, specifically of the type of Felder's Dynamical Quantum Groups [27] (see for instance [29, 28]). Whether doubly dynamical objects have such connections is a new problem and we have no further comments to make on this point. A very recent result [30] may however give indications on how to twist quantum groups by non-abelian twisted cocycles (here the twist would occur “along” a Heisenberg algebra).

3 The canonical Hamiltonians

3.1 Preliminaries

We first describe the Poisson-commuting Hamiltonians generated by traces of powers of the BC_N Lax matrix (3). They read for any integer $l \in \{1..N\}$:

$$\mathcal{H}_l = \text{tr}(L^l) = \sum_{\substack{\mathcal{J} \subset \{-N..N\} \\ |\mathcal{J}| = l}} m_{\mathcal{J}}(L),$$

with $m_{\mathcal{J}}(L)$ the principal minor of L with lines and columns indexed by \mathcal{J} .

Taking into account the form of L and properties of Cauchy matrices:

$$m_{\mathcal{J}}(L) = e^{-\beta\theta_{\mathcal{J}}} \prod_{\substack{j \in \mathcal{J} \\ k \notin \mathcal{J}}} f^{1/2}(q_j - q_k), \quad \text{where} \quad \theta_{\mathcal{J}} = \sum_{j \in \mathcal{J}} \theta_j.$$

Because of the folding, we now rearrange these terms so as to sort them with respect to linearly independent exponentials of rapidities. We thus decompose $\mathcal{J} = \varepsilon J \cup S$, separating indices of \mathcal{J} such that their opposite does not belong to \mathcal{J} (set $\varepsilon J = \{\varepsilon_j | j | j \in \mathcal{J} \wedge -j \notin \mathcal{J}\}$ and $J = \{|j| | j \in \varepsilon J\} \subset \{1..N\}$) and the complementary parts, symmetric under folding S :

$$\mathcal{H}_l = \sum_{\substack{J \subset \{1..N\}, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} \mathcal{U}_{J^c, l-|J|} e^{-\beta\theta_{\varepsilon J}} \prod_{\substack{j \in \varepsilon J \\ k \notin \varepsilon J}} f^{1/2}(q_j - q_k), \quad (15)$$

with

$$\mathcal{U}_{K,p} = \sum_{\substack{S \subset \mathcal{A}_K = K \cup -K \cup \{0\} \\ S = -S, |S| = p}} \prod_{\substack{s \in S \\ k \in \mathcal{A}_K \setminus S}} f^{1/2}(q_s - q_k) = \sum_{\substack{S \subset \mathcal{A}_K \\ S = -S, |S| = p}} \prod_{\substack{s \in S \\ k \in \mathcal{A}_K \setminus S}} v(q_s - q_k). \quad (16)$$

We then recall the Koornwinder-van Diejen Hamiltonians [14] in the classical case:

$$H_l = \sum_{\substack{J \subset \{1..N\}, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} U_{J^c, l-|J|} e^{-\beta\theta_{\varepsilon J}} V_{\varepsilon J; J^c}^{1/2} V_{-\varepsilon J; J^c}^{1/2}, \quad (17)$$

where, after some rearrangements,

$$V_{\varepsilon J; K} = \prod_{j \in \varepsilon J} \frac{w(q_j)}{v(2q_j)v(q_j)} \prod_{\substack{j \in \varepsilon J \\ k \in \mathcal{A}_K \cup -\varepsilon J}} v(q_j - q_k)$$

and

$$U_{K,p} = (-1)^p \sum_{\substack{\varepsilon I \subset \mathcal{A}_K \\ |I| = p}} \prod_{i \in \varepsilon I} \frac{w(q_i)}{v(2q_i)v(q_i)} \prod_{\substack{i, i' \in \varepsilon I \\ i < i'}} \frac{v(-q_i - q_{i'})}{v(q_i + q_{i'})} \prod_{\substack{i \in \varepsilon I \\ k \in \mathcal{A}_K \setminus \varepsilon I}} v(q_i - q_k).$$

The w are particular functions and may be interpreted as an interaction with some external field.

Direct computation yields:

$$V_{\varepsilon J; J^c} V_{-\varepsilon J; J^c} = \prod_{j \in \varepsilon J} \frac{w(q_j)}{v(2q_j)v(q_j)} \frac{w(-q_j)}{v(-2q_j)v(-q_j)} \prod_{\substack{j \in \varepsilon J \\ k \notin \varepsilon J}} f(q_j - q_k).$$

Setting $w(q_j) = v(2q_j)v(q_j)$, which is an admissible choice according to [14], H_l (17) takes actually the same form as \mathcal{H}_l (15), up to the change of $\mathcal{U}_{K,p}$ in to $U_{K,p}$. $U_{K,p}$ takes a simpler form, for this one body potential w :

$$U_{K,p} = (-1)^p \sum_{\substack{\varepsilon I \subset \mathcal{A}_K \\ |I| = p}} \prod_{\substack{i, i' \in \varepsilon I \\ i < i'}} \frac{v(-q_i - q_{i'})}{v(q_i + q_{i'})} \prod_{\substack{i \in \varepsilon I \\ k \in \mathcal{A}_K \setminus \varepsilon I}} v(q_i - q_k), \quad (18)$$

nevertheless *it is not generally equal to $\mathcal{U}_{K,p}$* (the notation used in [18] is in this respect misleading).

They are actually only equal for $p = 0$, where trivially: $U_{K,0} = 1 = \mathcal{U}_{K,0}$. For instance when $p = 1$, one gets:

$$U_{K,1} = - \sum_{i \in \mathcal{A}_K \setminus \{0\}} \prod_{\substack{k \in \mathcal{A}_K \\ k \neq i}} v(q_i - q_k) \quad \text{and} \quad \mathcal{U}_{K,1} = \prod_{\substack{k \in \mathcal{A}_K \\ k \neq 0}} v(q_k).$$

We compute a suitable contour integral on the same lines as in the previous section to obtain the Liouville-type functional identity:

$$\sum_{i \in \mathcal{A}_K} \prod_{\substack{k \in \mathcal{A}_K \\ k \neq i}} v(q_i - q_k) = \frac{\sinh \gamma (2|K| + 1)}{\sinh \gamma},$$

and thus show that: $U_{K,1} = \mathcal{U}_{K,1} - \frac{\sinh \gamma (2|K| + 1)}{\sinh \gamma}$.

It will now be shown that these two relations, for $p = 0$ and $p = 1$, between the $U_{K,p}$'s and $\mathcal{U}_{K,1}$'s are actually sufficient to establish that the two sets of Hamiltonians define the same family of commuting dynamical flows, namely one set of Hamiltonians is a triangular linear combination of the other set.

A more general result will in fact be proved in the following subsection.

3.2 Uniqueness theorem

Theorem 1 *Let q_i and θ_i , $i \in \mathbb{N}$, be a set of conjugated variables such that $\{\theta_i, q_j\} = \delta_{ij}$. Let I and K be arbitrary finite sets of indices included in \mathbb{N} . Assume the existence of a set of complex functions $u_{K,p}$ depending upon the set of indices K and a natural integer p , and of another set of complex functions $v_{\varepsilon J, I}$ depending upon the sets of indices J and I ($J \subset I$) and a $|J|$ -uple of signs $\varepsilon = (\varepsilon_j, j \in J)$, such that:*

- $u_{K,p}$ and $v_{\varepsilon J, I}$ be independent of the rapidities θ_i s.
- $u_{K,0} = 1$, $v_{\emptyset, I} = 1$, and $v_{\varepsilon\{j\}, I} \neq 0$.
- $S^I = \{ h_l^I = \sum_{\substack{J \subset I, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} u_{J^c, l-|J|} e^{-\beta \theta_{\varepsilon J}} v_{\varepsilon J, I}, l \in \{1..|I|\} \}$ be a family of Poisson-

commuting functions ($\theta_{\varepsilon J} = \sum_{j \in J} \varepsilon_j \theta_j$).

If there exists a second set of complex functions $\tilde{u}_{K,p}$ obeying the first two conditions; such that $\tilde{S}^I = \{ \tilde{h}_l^I = \sum_{\substack{J \subset I, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} \tilde{u}_{J^c, l-|J|} e^{-\beta \theta_{\varepsilon J}} v_{\varepsilon J, I}, l \in \{1..|I|\} \}$ be a new family of Poisson-commuting functions; and $\tilde{u}_{K,1} = u_{K,1} + c_1(|K|)$, then there exist coefficients $c_r(m)$, $(r, m) \in \mathbb{N}^2$, independent of all dynamical variables, connecting the two families of Hamiltonians as:

$$\tilde{h}_l^I = \sum_{s=0}^l c_{l-s}(|I| - s) h_s^I, \text{ with } \forall m \in \mathbb{N}, c_0(m) = 1.$$

Proof:

The strategy of the proof relies upon a recursive procedure on p , showing that:

$$\tilde{u}_{K,p} = \sum_{r=0}^p c_{p-r}(|K| - r) u_{K,r}, \quad \forall K \subset \mathbb{N} \text{ finite and such that } |K| \geq p. \quad (19)$$

Let l_0 be a strictly positive integer; the recursion hypothesis hereafter denoted r.h., states that (19) is valid for any $p \leq l_0$.

The assumptions in the theorem immediately imply the validity of r.h. for $l_0 = 1$, can be directly rewritten as:

$$\tilde{h}_1^I = h_1^I + c_1(|I|). \quad (20)$$

Let us assume r.h. up to l_0 and establish it for $l_0 + 1$. We have:

$$\tilde{h}_{l_0+1}^I = \tilde{u}_{I, l_0+1} + \sum_{\substack{J \subset I, 1 \leq |J| \leq l_0+1 \\ \varepsilon_j = \pm 1, j \in J}} \tilde{u}_{J^c, l_0+1-|J|} e^{-\beta \theta_{\varepsilon J}} v_{\varepsilon J, I}.$$

Since $l_0 + 1 - |J| \leq l_0$ in the previous summation, we apply r.h. to get:

$$\begin{aligned}
\tilde{h}_{l_0+1}^I &= \tilde{u}_{I,l_0+1} + \sum_{s=1}^{l_0+1} c_{l_0+1-s}(|I| - s) \sum_{\substack{J \subset I, 1 \leq |J| \leq s \\ \varepsilon_j = \pm 1, j \in J}} u_{J^c, s-|J|} e^{-\beta \theta_{\varepsilon J}} v_{\varepsilon J, I} \\
&= \tilde{u}_{I,l_0+1} - \sum_{s=1}^{l_0+1} c_{l_0+1-s}(|I| - s) u_{I,s} + \sum_{s=1}^{l_0+1} c_{l_0+1-s}(|I| - s) h_s^I. \tag{21}
\end{aligned}$$

We now use the Poisson-commutation property of \tilde{S}^I as:

$$\{\tilde{h}_{l_0+1}^I, \tilde{h}_1^I\} = 0.$$

Combining (20), (21) and the Poisson-commutation property of S^I yields:

$$\begin{aligned}
0 &= \{\tilde{u}_{I,l_0+1} - \sum_{s=1}^{l_0+1} c_{l_0+1-s}(|I| - s) u_{I,s}, h_1^I\} \\
&= \sum_{j \in I, \varepsilon = \pm 1} \beta \varepsilon e^{-\beta \varepsilon \theta_j} v_{\varepsilon \{j\}, I} \frac{\partial}{\partial q_j} (\tilde{u}_{I,l_0+1} - \sum_{s=1}^{l_0+1} c_{l_0+1-s}(|I| - s) u_{I,s}).
\end{aligned}$$

By functional independence of $\sum_{\varepsilon = \pm 1} \varepsilon e^{-\beta \varepsilon \theta_j} v_{\varepsilon \{j\}, I}$, it follows that the function ob-

tained as $\tilde{u}_{I,l_0+1} - \sum_{s=1}^{l_0+1} c_{l_0+1-s}(|I| - s) u_{I,s}$ *does not depend on any dynamical variable*.

Hence it defines the coefficient $c_{l_0+1}(|I|)$, thereby proving the r.h. to order $l_0 + 1$. Finally, relation (19) immediately implies the result of the theorem. \square

3.3 Comments

An immediate consequence of this theorem is the existence of linear triangular relations between the BC_N Ruijsenaars-Schneider Hamiltonians and the classical Koornwinder-van Diejen Hamiltonians when $w(q_j) = v(2q_j)v(q_j)$.

The explicit coefficient have to be computed order by order since at this time no general recursion formula is available. As an example we have worked out the first two functions:

$$U_{K,1} = \mathcal{U}_{K,1} + c_1(|K|) \quad \text{and} \quad U_{K,2} = \mathcal{U}_{K,2} + c_1(|K| - 1) \mathcal{U}_{K,1} + c_2(|K|)$$

$$\text{with} \quad c_1(|K|) = -\frac{\sinh \gamma (2|K| + 1)}{\sinh \gamma},$$

$$c_2(|K|) = \frac{1}{2} \left(\frac{\sinh \gamma (2|K| - 1)}{\sinh \gamma} \frac{\sinh \gamma (2|K| + 1)}{\sinh \gamma} - \frac{\sinh 4\gamma |K|}{\sinh 2\gamma} - 2 \right).$$

No obvious pattern appears yet. As a consequence, an algebraic interpretation of the Koornwinder-van Diejen Hamiltonians in connection with the canonical Hamiltonians is still lacking.

More generally, the theorem implies that a hierarchy of Poisson-commuting Hamiltonians with the generic form given is uniquely determined by the giving of the family of v -functions and the first Hamiltonian, or equivalently the first “potential term” $u_{K,1}$. In the Koornwinder-van Diejen case, this first Hamiltonian is given in [16].

We wish to end this section with a conjecture on the classical Koornwinder-van Diejen Hamiltonians with a general one-body potential chosen as in [14] (this time dropping the restriction to $w(q_j) = v(2q_j)v(q_j)$). They have not yet been constructed by a Lax formalism. We expect that the suitable Lax matrix for this hierarchy may be obtained by multiplying the BC_N Lax matrix (3) by a $2N + 1$ diagonal matrix: $L_W = LD_W$, with $D_{W_{ii}} = \mathcal{W}(q_i)\mathcal{W}(-q_i)$. This one-body potential function \mathcal{W} has to be determined by integrability conditions. In addition, we conjecture that, after some canonical transformation on the dynamical variables, the Hamiltonians $\mathcal{H}_W^l = \text{tr}(L_W^l)$ will take the same form as the Koornwinder-van Diejen Hamiltonians (17) up to the change of $U_{K,p}$ into some $\mathcal{U}_{W_{K,p}}$. The theorem will then apply, thereby yielding the full connection between BC_N -type Ruijsenaars-Schneider potentials and the classical Koornwinder-van Diejen Hamiltonians.

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